GUIDE #2

LEARNING IN UNDERGRADUATE MATHEMATICS:
THE OUTCOME SPECTRUM (LUMOS)
"HOW TO" GUIDES

Implement semi-authentic mathematical experiences

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The full list of titles in the series is:

"How to" Guide #1: Implement team-based learning

"How to" Guide #2: Implement semi-authentic mathematical experiences

"How to" Guide #3: Shift responsibility for learning onto students

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LEARNING IN UNDERGRADUATE MATHEMATICS: THE OUTCOME SPECTRUM (LUMOS).

"HOW TO" GUIDE #2:
IMPLEMENT SEMI-AUTHENTIC MATHEMATICAL EXPERIENCES

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A rationale for introducing semi-authentic mathematical experiences

It has been said that, in mathematics education, we spend 14 years (from ages 6-20) practising mathematics without actually "doing" mathematics in any real sense. While this overstates the case, and does a disservice to many teachers who work creatively with their classes, it is true that the explorative and creative aspects of mathematics are not a formal part of school or undergraduate curricula.

These characteristics of doing mathematics are not only appealing for many students, but they also give initial experiences that could build into good research practice. Many programmes fail to create opportunities for students to hypothesise, create and explore new mathematical objects, abstract and generalise, systematically examine consequences of definitions or assumptions, create counter-examples, and develop other essential skills (Cuoco, Goldenburg, & Mark, 1996). These skills are needed by research mathematicians, and by those employed in other fields for their mathematical expertise (Harel, Seldon & Seldon, 2006).

We undertook three trials of Engagement Sessions, a technique for introducing semi-authentic mathematical experiences to undergraduates. The feedback we received from students expressed strong appreciation for the opportunities to think differently about mathematics. A few took time to adjust to the new modes and expectations, but no students reported regret volunteering for these trials.:

"Years' worth of mathematics, being purely used to sculpt the perfect problem solver, pushed the function of creativity to the back of the mind: which is a shame because one of the main reasons I weaved mathematics into my life was for the sake of creativity. I'm going to miss this class when it's over and I have to go back to average maths" Student Feedback

During the LUMOS Project we asked lecturers what learning outputs they wanted for undergraduate students. They all mentioned mathematical processes in some form or another. However, no lecturers had "research behaviour" explicitly as an undergraduate aim. Moreover, lecturers have been heard to say that even Masters students are "not really ready for research". Yet our Engagement Sessions showed that many students were exhibiting beginning research behaviours, and that the explicit development of these at undergraduate level should be possible.

Many undergraduate programmes in the mathematical sciences, particularly those in the applied or professional fields, do include project work and other opportunities for semi-authentic mathematical activity. But how is it possible to incorporate such activities in large undergraduate classes?

What do we mean by semi-authentic mathematical experiences? Our intention was to give students experiences that more closely mirror what working mathematicians (researchers or those working mathematically in other fields) actually do. That is, we wanted students to explore new mathematical ideas, create new mathematical objects, argue rationally about these ideas, generalise, hypothesise, abstract, formalise, create examples and counter-examples, illustrate and explain mathematical ideas, get stuck and persist, self-evaluate their work, and a host of other mathematical behaviours.

Such activities were perceived as different from the usual tasks given to students at this level: learning conventional, formalised content and practising problem-solving skills.

An overview of this approach

"Just wanted to say something I didn't think to put on the feedback form, because it's become more apparent the last few days while I've been studying hard for the exam: I find that when I see a question I don't know how to answer, I'm finding it easier to just experiment with what I do know and use logic to figure out what the answer must be. I think I'm better at this due to the open-ended, exploratory nature of the engagement situations, and it's going to be very useful in the exam if I come across something I don't recognise"
Student Feedback

The approach described in this guide was designed for a large first year undergraduate standard mathematics course. It was trialled extensively on a course for students not intending to major in the mathematical sciences. We are confident that it would be useful for majoring students. We are aware that there are other approaches to giving students mathematical experiences that have also proven worthwhile.

This approach is a sequence of activities that replaces one or more standard assignments within a course. In our trials, we attempted four engagement sessions in a one-semester course, but this was too demanding. We succeeded in holding three sessions, but just one or two sessions would be effective.

The Engagement Session has three components:

- The setting of an open-ended mathematical situation for students to explore.
- Holding an Engagement Session—a small group tutorial with a lecturer.
- Writing a report.

Students are required to do some preliminary work on the mathematical situation, and come to the Engagement Session prepared to talk about what they have done. In the Engagement Session, the group listens to contributions, and then further explores the situation for a period of over one hour, starting from one or more of the ideas presented. Afterwards, students are expected to do further work following on from ideas discussed in the session. Finally, they write a 4-5-page report on all three aspects of the project. This is marked with about the same weight as an assignment.

Details of the approach

Setting the environment

A critical aspect of introducing mathematical activity is to ensure that the students understand what is being done, how it is assessed, and how it contributes (or not) to their overall grade. If expectations are misaligned, then students will quickly become dissatisfied.

Our experience was that, given a full explanation, many undergraduate students welcomed the opportunity to approach mathematics in a different way. They were happy to move away from the structured certainty of knowledge of specific curriculum topics and skills, and to engage with mathematics creatively.

However, many students did not initially understand exactly what they were meant to do. We recommend modelling mathematical exploration in front of the class and/or having a mini-trial of the student activity during a lecture, with guidance as to appropriate actions.

Part of setting the environment is also ensuring that students understand the value of the activity for their future work. Talking to them about the nature of research mathematics from a personal perspective is useful in this context.

See Appendix 1 for a sample information sheet for students.

Open-ended situations

Open-ended situations are intended to give students opportunities for doing creative mathematical thinking, rather than finding out the correct or generally accepted response. The emphasis is on original thinking and the exploration of the consequences of that thinking. The criteria we attempted to use were as follows:

- The topic should be unfamiliar to the students.
- The topic should be either difficult to research on the internet, or have many unresolved strands or alternative approaches.
- The topic should relate to, but be just beyond, the curriculum of the current course.
- The open-ended situation is given to the students about one week (including a weekend) before the Engagement Session is scheduled. Students are asked to undertake two hours' work exploring the situation, and be prepared to present their work to their group in the session.

See Appendix 2 for the open-ended situations we used with a standard first-year course.

Engagement Sessions

Engagement Sessions are meetings of 4-8 students with a lecturer. Eight students are an absolute maximum for effective results. Groups are selected by the lecturer, not by the students. The most effective way to do this is to announce possible session times, ask students to indicate all times they can attend, and then make up the groups.

The lecturer needs to be a confident and experienced teacher and mathematician. Casual tutors are not recommended for this role. Our meetings were scheduled for more than one hour, and usually lasted about one and a half hours. They never lasted more than two hours. It would be possible to hold them in a one-hour time-slot.

The meeting needs to be in a smallish space with tables for computers, and whiteboards at hand. Students are encouraged to bring computers, and it would be useful to have large sheets of paper and markers available. It is essential that the seating arrangement does not make the lecturer dominant—the atmosphere is intended to be one of mutual discussion/working together, not authority speaker/listener learner.

To start, two students are asked to spend 5-10 minutes presenting the work they have done on the situation. All students must be prepared to speak, and the speakers are chosen at the beginning of the meeting. On the first occasion, it is advisable to remind students about this the day before the meeting; after that it was not necessary.

Following the presentations, other students are invited to comment, say whether they did something similar, and expand further if they wish. This activity rarely goes on very long.

Next, the lecturer picks out one or two ideas from the presentations or discussions, possibly adds something or clarifies the idea, and then asks students to extend the idea further. A way to initiate activity at this point is to ask students to calculate or create further examples of the ideas presented.

At this point the meeting may take different forms. Behaviour that we have experienced includes:

- Discussion and argumentation by the whole group on one idea.
- Different students pursuing different ideas, regularly reporting on progress.
- Some students doing internet searches or using mathematical environments for drawing graphs or making calculations.
- Lecturer taking "time out" to alert students to particularly good examples of good mathematical habits, such as persistence, hypothesising, exploring consequences, etc.
- If the discussion peters out or reaches a dead end, it may be necessary for the lecturer to reset the group using another idea that has emerged from the student presentations.

Towards the end of the session, the lecturer needs to comment on and make a summary of the progress made on each idea, pointing out areas that students could focus on for their post-session work. Students may make notes as preparation for writing their reports. Students may also take cell-phone pictures of whiteboard or paper workings from the group activity. This is also a good opportunity to praise instances of good mathematical behaviour.

Reports

After the Engagement Session students are expected to spend another two hours on further exploration.

Then they should write a 4-5-page report including:

- A short account of their pre-session work (1 page).
- A short account of the Engagement Session (1-2 pages).
- An account of their post-session work (1-2 pages).

These reports are likely to be worth about the same as an assignment.

The reports are marked by the lecturer, giving a single mark out of 5, focusing on the quality of the reporting (coherence, coverage, and correctness of mathematical expression) and creativity/originality of the work, rather than the sophistication of the mathematics.

Issues to be aware of

Economies of scale

A possible reason that this sort of activity is not undertaken in large classes is the time requirement on lecturing staff. We calculate that holding 1-2 hour sessions with groups of eight students, plus marking reports, for large classes of up to 2-300 students will be very time-consuming. Our trials were all with classes under twenty students, but we are confident that the time required can be managed with classes up to 150 students.

In our trials, we ameliorated these time demands by combining the Engagement Sessions with three other course innovations:

- The number of lectures per week was reduced from three to one by using on-line tutorials (e.g. Khan Academy) to teach basic content and skills.
- Some of the standard once-per-week tutorials were made voluntary.
- Marking of mid-semester tests and examinations was eliminated by using computer-marked short-answer questions.

In order to address the time demands mentioned above, we recommend that:

- The first trial is for a cohort of less than 150 students.
- Two lecturers are allocated to the task.
- In the week of the Engagement Session all lectures and tutorials are replaced by set work using web-based resources.
- The Engagement Session is held over one hour only.
- Reports are marked very quickly with a single mark and no further feedback is given (3-4 minutes per report maximum) (Note: this is 10 hours of marking for 150 reports).

After a few such experiences, it will be possible to make a sensible judgment, weighing positive outcomes with resource requirements, for the context in which your undergraduate courses are run.

It is possible that tutors from PhD level, with good teaching skills, could be used in the Engagement Sessions, however, we would recommend that sessions are undertaken by lecturers in the first instance, and the less experienced tutors can then observe the session before taking a session themselves. The reason for this is that the Engagement Sessions require a high level of teacher skills and mathematical “thinking-on-your-feet” (see the section on Teaching Demands over page →).

Marking

Some of the economy of scale can be recovered by efficient marking, so that the marking time is significantly shorter than that required for the assignment that has been replaced. Our experience was that a single mark out of 5 was easy to make after a single reading of the report. Furthermore, this mark was satisfactory as far as the students were concerned, particularly if a single additional feedback comment is also made. In many instances, this feedback comment was of the kind: "If you would like to follow your ideas further then a Google search on xxx may be of interest".

The lack of detailed feedback is a lost opportunity for formative assessment, thus some balanced response is required. In our trial, we achieved this by giving verbal feedback on the reports to the class as a whole—many of the issues were common to many students. If the class numbers are low, or other time pressure is absent, then some written feedback is possible.

As noted above, the report is not a place to evaluate mathematical content knowledge. Rather, it is a place to evaluate mathematical habits or processes. Indeed, this is an added advantage of Engagement Sessions, as many existing courses have few opportunities to focus student attention on processes. Some aspects that can be part of the report evaluation are as follows.

- Mathematical communication: Is the report written using appropriate and correct mathematical notation, appropriate diagrams or graphs, and explanatory text?
- Completeness and coherence: Does the report "tell a story" of an exploration undertaken in three stages?
- Mathematical creativity and originality: Are the ideas being explored new (for the student), and formulated in a mathematical way?
- Follow-through: Are the ideas examined for their rationality and consequences, both internally and with respect to other aspects of mathematics?

Student Time

The opportunity to be explorative and creative can be quite addictive for some students. We found that many of our students got carried away with their ideas. For example, one student working on functions from **R** to **R** x **R** started conceiving musical chords as such functions, and began to play with the idea of adding, subtracting and composing such functions. His report included his own musical files.

In terms of motivation, such work was valuable, but care needs to be taken that students do not invest more time than is sensible. This particularly applied to report writing where many students wrote reports that were too long (and therefore also required more marking time).

Teaching Demands

"Wow, that was much harder than [I expected]".
Comment after a lecturer's first Engagement Session.

The task of running an engagement session is very different from both conventional lecturing and tutoring.

It is unlike a lecture because, beyond the general area of focus, the ideas that arise and need to be managed and discussed are not known in advance. Considerable awareness and "thinking-on-your-feet" is required in order to hear and identify ideas that are liable to be productive, and sensitivity is required in order to indicate and encourage productive discussion. The situation is new for students, and they need support and acknowledgement of their contributions.

The Engagement Session is also unlike a conventional tutorial. Students are not working on mathematics with known or "correct" answers. It is not their understanding that requires facilitating, it is their mathematical thinking and mathematical habits. While working together in groups may be familiar for students, working together creatively is not.

In order to prepare for an Engagement Session for the first time, a lecturer could try:

- "playing" with the situation themselves;
- discussing the situation with other lecturers or graduate students;
- finding out some background information on the students (for example, their majors, their mathematical background, NCEA or other examination results or previous grades);
- researching links between the situation and other mathematical topics or applications.

References

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Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In *Handbook of Research on the Psychology of Mathematics Education*, A. Gutierrez and P. Boero, (Eds.), Sense Publishers, Rotterdam, Netherlands, pp. 147–172.

Appendix 1

Sample Engagement Session Information Sheet

Introduction to Engagement Sessions

Engagement Sessions are a new activity that are designed to give students the experience of working mathematically in a way similar to the way people work outside education; both mathematical researchers, and those using mathematics to solve real world problems.

In this course, there will be three Engagement Sessions—you are expected to participate in all three. They will take place in weeks 4, 8, and 11. Each Engagement Session will result in a Report of four to five pages. This will be marked and the best two marks will become your Assignment Mark for the course. You may also complete the Assignment and have it marked if you wish in order to get more feedback about your work.

Each Engagement Session will have three parts.

First, in the weekend before the Session, you will be given an open-ended situation to explore mathematically. You are expected to work for about two hours on this situation, using whatever resources you wish. Be prepared to speak for five to ten minutes about what you have done.

Second, we will meet for an hour and a half in a group of six to eight students with the lecturer. The groups will be chosen randomly, or possibly based on when people are available at the same time. I will ask two people to speak about the work they have done. The two people will not know who they are until the Engagement Session starts. We will then all work together cooperatively using the work presented by the two speakers and also any other work done beforehand. We will develop what has been done, look for new questions to ask, link the work to what we have been learning, and so on.

Third, each student will go away, and do some more work—up to an hour—using whatever resources they wish. They will then write this up into a report of four to five pages, and hand it in.

The report will be marked and feedback given.

Appendix 2

Sample Open-ended situations

These situations were given to a non-advancing mathematics class whose students had completed a Year 13 Mathematics course successfully. (The first and last situations were the most successful).

SAMPLE 1

Functions from R to R × R

Most functions we have been using so far in your course map a Real Number onto a Real Number. We write f: **R** → **R**, and we say "f maps **R** onto **R**".

But functions can be about any numbers, not necessarily the Real Numbers: Whole Numbers, Complex Numbers, many different sorts of numbers. That is why we have to specify the domain when we define a function. In fact, they do not have to be numbers at all, a function can map anything onto anything. In this course, you will find functions involving vectors and matrices, for example.

But not only that, we can define functions that map TWO numbers onto one number. You will learn more about such functions at the end of this course—they are also called multivariate functions, because they have more than one variable. An example of such a function is

$$f(x, y) = 3x - y^2$$

Thus, we start with an x-value and a y-value, say, *1* and *2*, and get another value, in this case

1. We write f: **R** × **R** → **R**, and we say "f maps **R** cross **R** onto **R**".

Now think about a function that works the other way. It starts with a single Real Number but produces TWO Real Numbers. That is f: **R** → **R** × **R**

Our first problem is to find a suitable notation. Let's take an example. We start with a function f and a variable x. Let the first number created by the function be x^2, and the second number be $(1/x)$. Thus $f(2)$ is both 4 and ½.

Here is your first task: Devise a suitable notation for this.

Your second task is to devise a new function h: $\mathbf{R} \to \mathbf{R} \times \mathbf{R}$. That is, make up your own rules for h. Explore some values of h. Is h a function? That is, will each separate input x give a unique output pair?

Can you find a way of graphing h? This will need to be a new kind of graph.

What can you say about the values of h for different inputs? E.g. what happens to $h(x)$ when x is close to zero, when x gets very large, when x is negative, when … find some other things to investigate about $h(x)$.

Can you find another function, $j(x)$, which behaves differently? Will your graphing and notation scheme work for $j(x)$?

Be ready to talk about what you have done in your Engagement Session. You will have a whiteboard or computer display if you need it.

SAMPLE 2

Picturing Spaces

In this course, we mostly deal with 2- or 3-dimensional objects and 2- or 3-dimensional spaces. In this discussion, we would like to ask you to think about higher-dimensional spaces.

A 2-dimensional object is, for example, a circle. A 2-dimensional space is the area in which the circle exists, for example, a plane piece of paper.

A 3-dimensional object is, for example, a sphere. A 3-dimensional space is the area in which the sphere exists, for example, the inside of a room.

Generally, the intersection of two lines is a point, and the intersection of two planes is a line (except in the special case where the planes are coincident, or they are parallel).

What happens if you take two objects (surfaces) in 3-D space that are not planes, and intersect them? E.g., what happens if you intersect the surfaces of two spheres? Try some other objects.

What is the intersection of two 3-D spaces? Can you draw or make a model of this?

SAMPLE 3

Transforming Shapes

Matrices have many uses. For example, a 2×2 matrix can be thought of as a transformation—i.e., as a way of moving a vector. If you multiply a 2×1 vector by a 2×2 matrix you get another 2×1 vector.

NOTE: Please familiarise yourself with how to add, subtract, and multiply matrices before attempting to work on this situation.

E.g. The matrix $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

transforms the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Some matrices represent particular familiar transformations with EVERY vector:

E.g. The matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

reflects every vector in the y-axis, e.g.: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Try this with a couple of other vectors.
Can you find other matrices which represent particular transformations, e.g., reflection in the x-axis, or half turn rotation?

HINT: Consider what you want to happen to the two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The beauty of mathematics is that we can use patterns to find results in higher dimensions. 3×3 matrices can be thought of as transformations in 3-space. Can you create the matrix that will always reflect a vector in the xy-plane?

HINT: Consider what you want to happen to the three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

What else can you do with matrices as geometric transformations? For instance, can you find a matrix that transforms squares into parallelograms?

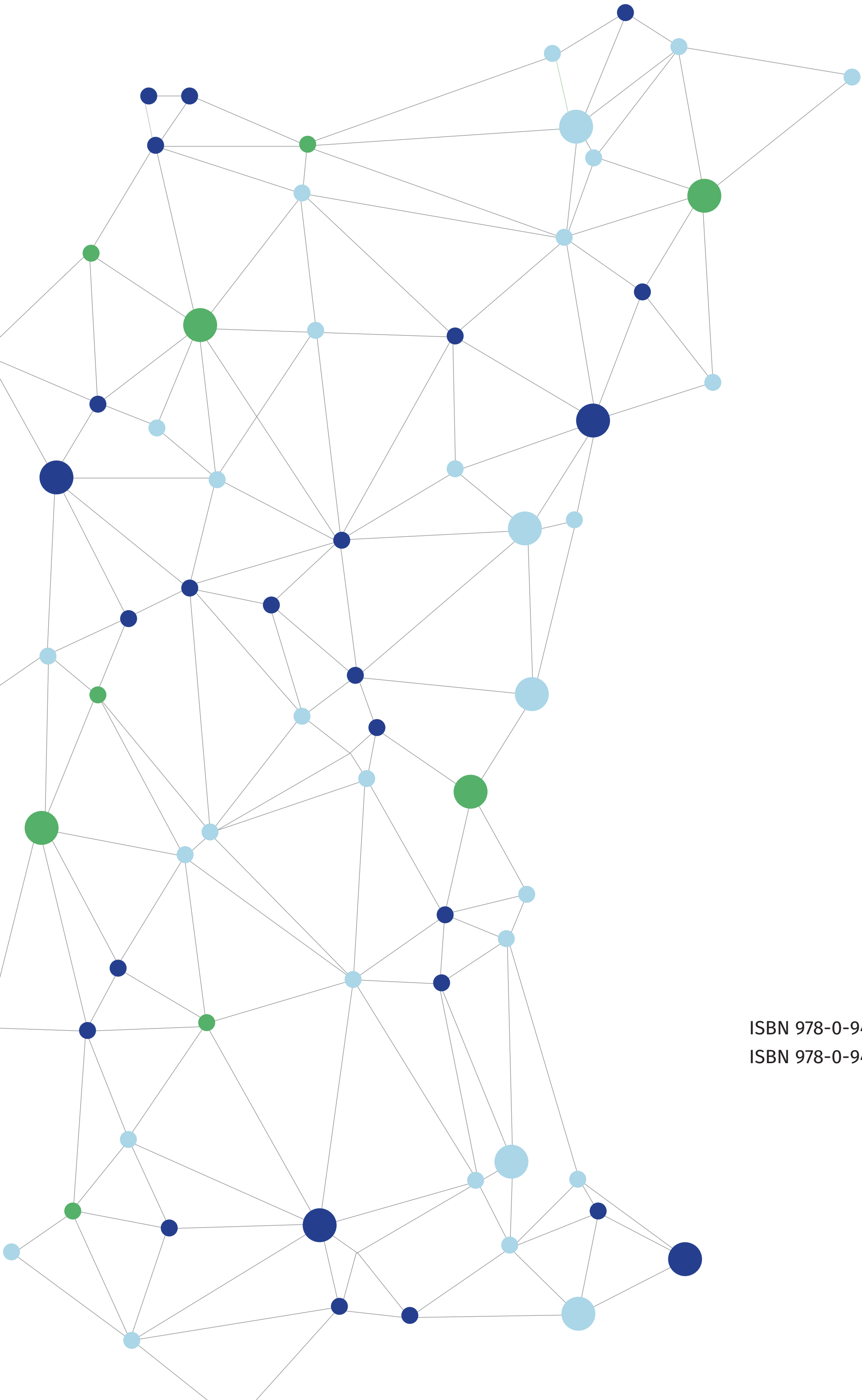
SAMPLE 4

Fractional Calculus

You will have used the first derivative of a function to locate maxima and minima, and the second derivative to determine curvature. What use might the third derivative be? Or why might we continue taking higher and higher derivatives?

What about the zeroth derivative? What might that be? Or the minus oneth derivative?

Can you give a sensible meaning to fractional derivatives? How might the halfth derivative be defined? Can you give it a meaning?



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